Integrals in multicriteria decision support

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In multicriteria decision making with n criteria, $N = \{1, \ldots, n\}$, normalized compatible scales $[0, 1]$, $m : 2^N \to [0, 1]$

"boolean utility"

$$m(\emptyset) = 0, \ m(N) = 1,$$

$$E \subset F \Rightarrow m(E) \leq M(F)$$

m capacity, fuzzy measure, weights of criteria groups, monotone game ...

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aim: extend m to graded utility function

$$I_m : [0,1]^n \to [0,1]$$

unanymous (idempotent)
nondecreasing (Pareto)
$I_m(c\,1_E)$ depends on c and $m(E)$ only!
then INTEGRAL
Klement, Mesiar, Pap ≈ Universal integrals ≈2010 in IEEE TFS

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$$I_m(c\,1_E) = c \otimes m(E)$$

$$\otimes : [0,1]^2 \to [0,1] \quad \text{semicopula}$$

the **weakest universal** integral

$$I_m^{\otimes}(\mathbf{x}) = \bigvee_i \left(x_i \otimes m(\{j \mid x_j \geq x_i\})\right)$$

$I_M^{\otimes}$ **SUGENO** integral

$I_\Pi^{\otimes}$ **SHILKRET** integral

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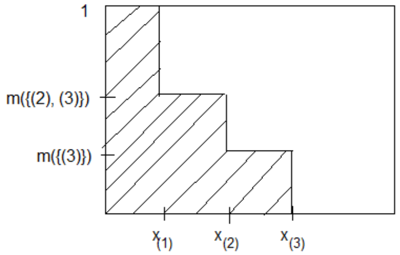
$$C : [0,1]^2 \to [0,1] \text{ a fixed copula}$$

$$I_{C,m}(\mathbf{x}) = P_C\left(\{(x,y) \in [0,1]^2 |\, y \le m(\{i \in N |\, x_i \ge x\})\}\right)$$



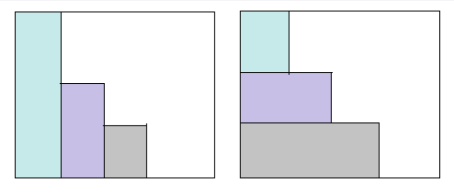
$(\cdot) : N \to N$ a permutation

$$x_{(1)} \le x_{(2)} \le \cdots \le x_{(n)}$$

$$I_{C,m}(\mathbf{x}) = \sum_{i=1}^{n} \left(C(x_{(i)}, m(\{(i), \dots, (n)\}) - C(x_{(i-1)}, m(\{(i), \dots, (n)\})\right) =$$

$$= \sum_{i=1}^{n} \left(C(x_{(i)}, m(\{(i), \dots, (n)\}) - C(x_{(i)}, m(\{(i+1), \dots, (n)\})\right)$$

$$x_{(0)} \equiv 0, \quad m(\{(n+1), (n)\}) \equiv 0$$

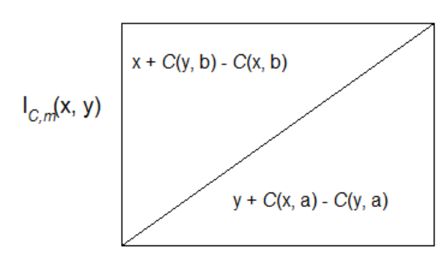


$I_{\Pi,m}$ CHOQUET integral
$I_{M,m}$ SUGENO integral

$$I_{C,m}(t \cdot 1_E) = C(t,\, m(E))$$

C expresses the dependence between function values and measure values

$$n = 2, \quad m(\{1\}) = a, \quad m(\{2\}) = b, \ a, b \in [0, 1]$$



m symmetric
= depends only on number of considered criteria,

$$|E| = |F| \Rightarrow m(E) = m(F)$$

$I_{C,m}$ anonymous (symmetric)
$I_{C,m} \equiv OMA$
Ordered Modular Average
comonotone modular, symmetric and idempotent aggregation function
Mesiar & Zemánková, IEEE TFS

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$(w_1, \ldots, w_n)$ discrete probability vector,

$$m(E) = \sum_{i=1}^{n} w_{n-i+1} = v_{n-|E|+1}$$

$$I_{\Pi,m}(\mathbf{x}) = \sum_{i=1}^{n} x_{(i)} w_i$$

OWA Yager 1988

$$I_{M,m}(\mathbf{x}) = \bigvee_{i=1}^{n} (x_{(i)} \wedge v_i)$$

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In general, for a symmetric m,

$$I_{C,m}(\mathbf{x}) = \sum_{i=1}^{n} \big(C(x_{(i)}, v_i) - C(x_{(i)}, v_{i+1})\big) = \sum_{i=1}^{n} f_i(x_{(i)}) = \sum_{i=1}^{n} w_i \cdot g_i(x_{(i)})$$

$$f_i : [0,1] \to [0,1], \text{ nondecreasing, } \textstyle\sum_{i=1}^{n} f_i = id$$
$$g_i : [0,1] \to [0,1], \text{ nondecreasing surjection}$$

$$w_i \cdot g_i = f_i$$

convention $v_{n+1} = 0$

$$n = 2 \quad (w_1, w_2)$$

$$I_{C,m}(x_1, x_2) = C(x_{(1)}, 1) - C(x_{(1)}, w_2) + C(x_{(2)}, w_2) - C(x_{(2)}, 0) =$$

$$= \left(x_{(1)} - C(x_{(1)}, w_2)\right) + C(x_{(2)}, w_2)$$

$$I_{W,m}(x_1, x_2) = f_1(x_{(1)}) + f_2(x_{(2)})$$

$$f_1(x) = min\,\{x, w_1\}$$

$$f_2(x) = max\,\{0, x - w_1\}$$

$$f_1(x) + f_2(x) = x$$

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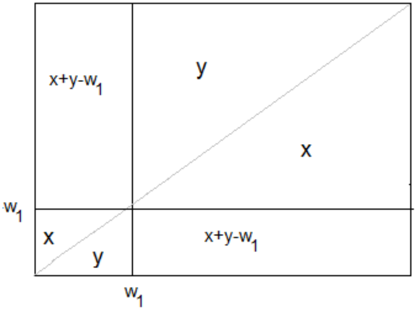
$I_{W,m}(x, y)$



Thanks for your attention