Integrals in multicriteria decision support

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In multicriteria decision making with n criteria, $N = \{1, \ldots, n\}$, normalized compatible scales $[0, 1]$, $m : 2^N \to [0, 1]$

"boolean utility"

$$m(\emptyset) = 0, \ m(N) = 1,$$

$$E \subset F \Rightarrow m(E) \leq M(F)$$

m capacity, fuzzy measure, weights of criteria groups, monotone game ...

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aim: extend m to graded utility function

$$I_m : [0, 1]^n \to [0, 1]$$

unanymous (idempotent)
nondecreasing (Pareto)
$I_m(c\,1_E)$ depends on c and $m(E)$ only!
then INTEGRAL
Klement, Mesiar, Pap $\approx$ Universal integrals $\approx$ 2010 in IEEE TFS

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$$I_m(c\,1_E) = c \otimes m(E)$$

$$\otimes : [0,1]^2 \rightarrow [0,1] \quad \text{semicopula}$$

the **weakest universal** integral

$$I_m^{\otimes}(\mathbf{x}) = \bigvee_i \left(x_i \otimes m\left(\{j \,|\, x_j \geq x_i\}\right)\right)$$

$I_M^{\otimes}$	**SUGENO** integral
$I_\Pi^{\otimes}$	**SHILKRET** integral

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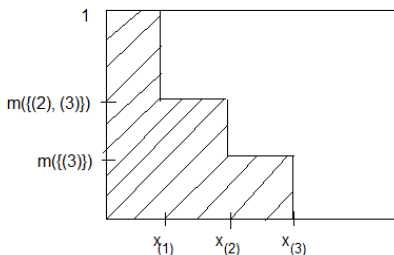
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$$C : [0, 1]^2 \rightarrow [0, 1] \text{ a fixed copula}$$

$$I_{C,m}(\mathbf{x}) = P_C \left(\{ (x, y) \in [0, 1]^2 | y \leq m(\{ i \in N | x_i \geq x \}) \} \right)$$

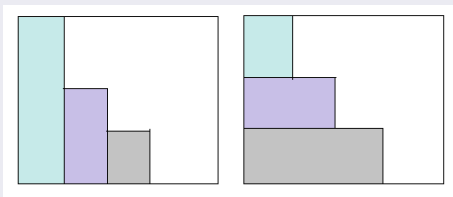$$(\cdot) : N \rightarrow N \text{ a permutation}$$

$$x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(n)}$$

$$I_{C,m}(\mathbf{x}) = \sum_{i=1}^{n} \left(C(x_{(i)}, m(\{(i), \ldots, (n)\})) - C(x_{(i-1)}, m(\{(i), \ldots, (n)\})) \right) =$$

$$= \sum_{i=1}^{n} \left(C(x_{(i)}, m(\{(i), \ldots, (n)\})) - C(x_{(i)}, m(\{(i+1), \ldots, (n)\})) \right)$$

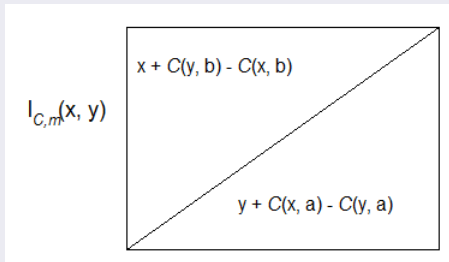$$x_{(0)} \equiv 0, \quad m(\{(n+1), (n)\}) \equiv 0$$

$I_{\Pi,m}$ CHOQUET integral
$I_{M,m}$ SUGENO integral

$$I_{C,m}(t \cdot 1_E) = C(t, m(E))$$

C expresses the dependence between function values and measure values

$$n = 2, \quad m(\{1\}) = a, \quad m(\{2\}) = b, \ a, b \in [0, 1]$$

$I_{C,m}(x, y)$

x + C(y, b) - C(x, b)

y + C(x, a) - C(y, a)

m symmetric
= depends only on number of considered criteria,

$$|E| = |F| \Rightarrow m(E) = m(F)$$

$I_{C,m}$ anonymous (symmetric)
$I_{C,m} \equiv OMA$
Ordered Modular Average
comonotone modular, symmetric and idempotent aggregation
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Mesiar & Zemánková, IEEE TFS

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$(w_1, \ldots, w_n)$ discrete probability vector,

$$m(E) = \sum_{i=1}^{n} w_{n-i+1} = v_{n-|E|+1}$$

$$I_{\Pi,m}(\mathbf{x}) = \sum_{i=1}^{n} x_{(i)} w_i$$

OWA Yager 1988

$$I_{M,m}(\mathbf{x}) = \bigvee_{i=1}^{n} (x_{(i)} \wedge v_i)$$

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In general, for a symmetric m,

$$I_{C,m}(\mathbf{x}) = \sum_{i=1}^{n} \left(C(x_{(i)}, v_i) - C(x_{(i)}, v_{i+1}) \right) = \sum_{i=1}^{n} f_i(x_{(i)}) = \sum_{i=1}^{n} w_i \cdot g_i(x_{(i)})$$

$f_i : [0,1] \to [0,1]$, nondecreasing, $\sum_{i=1}^{n} f_i = id$
$g_i : [0,1] \to [0,1]$, nondecreasing surjection

$$w_i \cdot g_i = f_i$$

convention $v_{n+1} = 0$

$$n = 2 \quad (w_1, w_2)$$

$$I_{C,m}(x_1, x_2) = C(x_{(1)}, 1) - C(x_{(1)}, w_2) + C(x_{(2)}, w_2) - C(x_{(2)}, 0) =$$

$$= (x_{(1)} - C(x_{(1)}, w_2)) + C(x_{(2)}, w_2)$$

$$I_{W,m}(x_1, x_2) = f_1(x_{(1)}) + f_2(x_{(2)})$$

$$f_1(x) = min\{x, w_1\}$$

$$f_2(x) = max\{0, x - w_1\}$$

$$f_1(x) + f_2(x) = x$$

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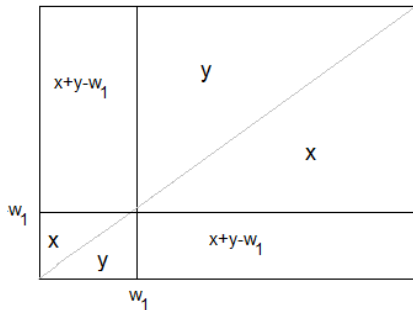
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$I_{W,m}(x, y)$

Thanks for your attention