Copulas

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Introduction

Abe Sklar 1957

copula $C : [0,\ 1]^n \to [0,\ 1]$

Sklar theorem

$$Z = (X_1, \ldots, X_n)$$

$$F_Z(x_1, \ldots, x_n) = C\left(F_1(x_1), \ldots, F_n(x_n)\right)$$

$$X_i \sim \text{ uniform on }]0, 1[$$

$$C \equiv F_Z \,|\, [0, 1]^n$$

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$$C \equiv F_Z \,|\, [0,1]^n$$

Axiomatic approach

i) $C(x_1, \ldots, x_n) = 0$ if some $x_i = 0$
C is **grounded**

ii) $C(x_1, \ldots, x_n) = x_i$ if $\forall j \neq i, x_j = 1$
C has **neutral element 1**

iii) $\forall u_1 \leq v_1, \ldots, u_n \leq v_n$
$\sum (-1)^{|I|} C(t_1^I, \ldots, t_n^I) \geq 0$
$I \subseteq \{1, \ldots, n\} \quad t_i^I = \begin{cases} u_i & \text{if } i \in I, \\ v_i & \text{if } i \notin I \end{cases}$
C is n–**increasing**

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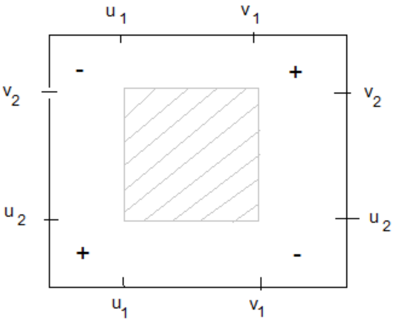


Each C is **1–Lipschitz** (wrt. L_1–norm)

$$C(v_1, v_2) - C(v_1, u_2) - C(v_2, u_1) + C(u_1, u_2) \geq 0$$

2–increasing, supermodular

$$W \leq C \leq M$$

$$M(u_1, \ldots, u_n) = min\ \{u_1, \ldots, u_n\}$$

$$W(u_1, \ldots, u_n) = max\ \left\{0, \sum_{i=1}^{n} u_i - n + 1\right\}$$

$$\Pi(u_1, \ldots, u_n) = \prod_{i=1}^{n} u_i$$

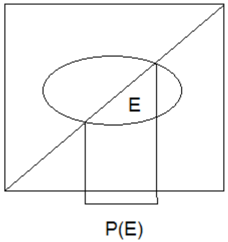
$\mathcal{C}_n$ class of all n–ary copulas is convex and compact

Example

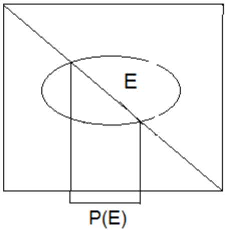
Copulas are in 1–1 correspondence with probability measures on $\mathcal{B}([0,1]^n)$ with uniform 1–dimensional marginals

$$\Pi \sim \text{Lebesgue}$$

M



W



Statistical interpretation

$$\Pi \sim \text{ independence}$$

$$M \sim \text{ comonotone dependence,}$$

$$X_i = f_i(X_1), \quad f_i \nearrow$$

$$W \sim \text{ counter monotone dependence,}$$

$$X_2 = g(X_1), \quad g \searrow$$

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Gaussian and t–copulas $\sim$ eliptic copulas, no closed form

$$C(u_1, \ldots, u_n) = F_Z\left(F_1^{-1}(u_1), \ldots, F_n^{-1}(u_n)\right)$$

X_i continuous, $F_i(X_i)$ uniform on $]0, 1[$

in general, margins need not be Gaussian (Student)!

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flipping

$$C^{-}(u,v) = u - C(u, 1-v)$$

$$(X,Y) \to (X,-Y)$$

$$C_{-}(u,v) = v - C(1-u, v)$$

$$(X,Y) \to (-X,Y)$$

survival

$$\hat{C}(u,v) = u + v - 1 + C(1-u, 1-v)$$

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Ordinal sums

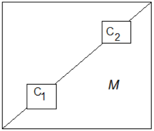

Figure: *M*–ordinal

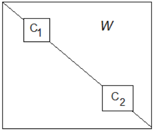

Figure: *W*–ordinal

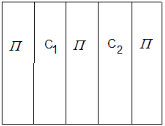

Figure: Π–ordinal

generators $f : [0,1] \to [0,\infty]$
strictly decreasing, continuous,

$$f(1) = 0$$

$$C_f(u_1, \ldots, u_n) = f^{-1}\left(min\left\{ f(0), \sum_{i=1}^{n} f(u_i) \right\}\right)$$

$$g : [-\infty, 0] \to [0,1], \quad g(x) = f^{-1}\left(min\left\{f(0), -x\right\}\right)$$

$$g' \geq 0, \ldots, g^{(n-2)} \geq 0, \quad g^{(n-2)} \text{ convex}$$

$$n = 2 \;\equiv\; f \text{ convex}$$

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Archimedean copulas

associative and $C_f(u, \ldots, u) < u$ for $u \in]0, 1[$

$$f_\Pi(x) = -log\, x \quad \forall n$$

$$f_W(x) = 1 - x \quad \text{only } n = 2$$

n fixed, weakest $C_{f^{[n]}}$

Clayton copula

$$f^{[n]}(x) = 1 - x^{\frac{1}{n-1}}$$

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$$f_\lambda^C(x) = \frac{x^{-\lambda} - 1}{\lambda}, \quad \lambda \in [-1, \infty[, \quad \lambda \neq 0$$

$$f_0^\lambda = f_\Pi, \quad \lambda \geq 0 \forall n$$

$\lambda = 1$ ALI–MIKHAIL–HAQ copula

Hamacher product

$$C(u, v) = \frac{uv}{u + v - uv}$$

Gumbel

$$\lambda \in [1, \infty[\quad f_\lambda^G(x) = (-log\, x)^\lambda, \quad \forall n$$

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$$C_{(f)}(u,v) = u\, f^{-1}\left(min\left\{ \frac{f(v)}{u},\, f(0) \right\} \right)$$

Univariate Conditioning Stable

$$C_{(f_1^C)}(u,v) = \frac{u^2\, v}{1 - u + uv}$$

$$C_{(f_\Pi)}(u,v) = u\, v^{-\frac{1}{u}}$$

$$C_{(f_W)} = W$$

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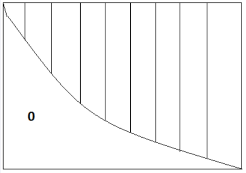
Associative copula $\equiv$ M–ordinal sums of Archimedean

0 - y - semilinear copulas



$$v = 1 - d(u)$$
$$C_d(u,v) = u \cdot \max\left\{0, 1 + \frac{v-1}{d(u)}\right\}$$
$$\text{distortion } d : [0,1] \to [0,1]$$
$$d(x) \nearrow, \quad \frac{x}{d(x)} \nearrow, \quad d(1) = 1$$

$$C_{id} = W, \quad C_1 = \Pi$$

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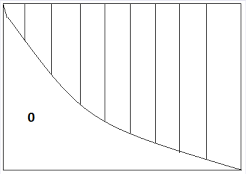

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DUCS copulas
Distorted **U**nivariate **C**onditioning **S**table

$$C_{f,d}(u,v) = u \cdot f^{-1}\left(\min\left\{f(0), \frac{f(v)}{d(u)}\right\}\right) \quad u > 0$$

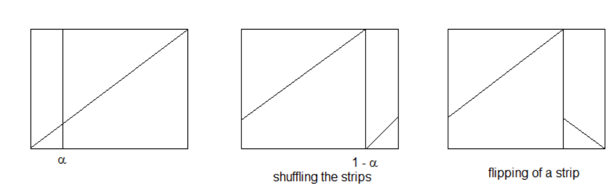
$$d(x) = x^p, \quad p \in]0,1]$$

$$f(x) = \frac{1}{x} - 1$$

$$C_{f,d}(u,v) = \frac{u^{p+1}v}{1 + u^p v - v}$$

Special singular copulas

Shuffles of M



Each copula can be seen as a limit of shuffles of M

Extreme values (EV) copulas

$$(X_1, Y_1) \sim C$$

$$\vdots$$

$$(X_n, Y_n) \sim C$$

$$\left(\bigvee X_i, \bigvee Y_i\right) \sim C_n$$

$$C_n(u, v) = \left(C(u^{\frac{1}{n}}, v^{\frac{1}{n}})\right)^n \xrightarrow[n \to \infty]{} C^*(u, v)$$

$$C^*\left(u^\lambda, v^\lambda\right) = \left(C^*(u, v)\right)^\lambda$$

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$$C^*(u,v) = (u\,v)^{A\left(\frac{\log\ u}{\log\ uv}\right)} =$$

$$= g^{-1}\left(\min\left\{g(0), (g(u)+g(v))\right) \cdot A\left(\frac{g(u)}{g(u)+g(v)}\right\}\right)$$

$$g(x) = -\log x$$

$$A : [0,1] \to [0,1] \quad \text{convex}$$

$$t \vee (1-t) \le A(t) \le 1$$

Archimax

$$C_{f,A}(u,v) = f^{-1}\left(\min\left(f(0), (f(u)+f(v))\right) \cdot A\left(\frac{f(u)}{f(u)+f(v)}\right)\right)$$

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Thanks for your attention