

Copulas

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Introduction

Abe Sklar 1957

copula $C : [0, 1]^n \rightarrow [0, 1]$

Sklar theorem

$$Z = (X_1, \dots, X_n)$$

$$F_Z(x_1, \dots, x_n) = C(F_1(x_1), \dots, F_n(x_n))$$

$X_i \sim$ uniform on $]0, 1[$

$$C \equiv F_Z | [0, 1]^n$$

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Axiomatic approach

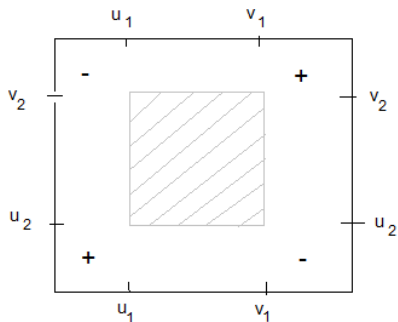
- i) $C(x_1, \dots, x_n) = 0$ if some $x_i = 0$
 C is **grounded**
- ii) $C(x_1, \dots, x_n) = x_i$ if $\forall j \neq i, x_j = 1$
 C has **neutral element 1**
- iii) $\forall u_1 \leq v_1, \dots, u_n \leq v_n$
 $\sum (-1)^{|I|} C(t_1^I, \dots, t_n^I) \geq 0$
 $I \subseteq \{1, \dots, n\} \quad t_i^I = \begin{cases} u_i & \text{if } i \in I, \\ v_i & \text{if } i \notin I \end{cases}$
 C is **n-increasing**

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Each C is **1-Lipschitz** (wrt. L_1 -norm)

$$C(v_1, v_2) - C(v_1, u_2) - C(v_2, u_1) + C(u_1, u_2) \geq 0$$

2-increasing, supermodular

$$W \leq C \leq M$$

$$M(u_1, \dots, u_n) = \min \{u_1, \dots, u_n\}$$

$$W(u_1, \dots, u_n) = \max \left\{ 0, \sum_{i=1}^n u_i - n + 1 \right\}$$

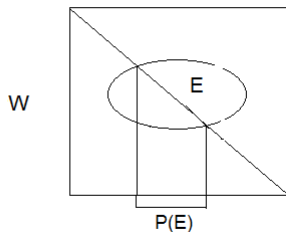
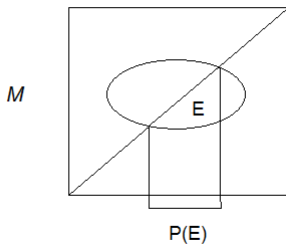
$$\Pi(u_1, \dots, u_n) = \prod_{i=1}^n u_i$$

\mathcal{C}_n class of all n -ary copulas is convex and compact

Example

Copulas are in 1–1 correspondence with probability measures on $\mathcal{B}([0, 1]^n)$ with uniform 1–dimensional marginals

$$\Pi \sim \text{Lebesgue}$$



Statistical interpretation

$\Pi \sim$ independence

$M \sim$ comonotone dependence,

$$X_i = f_i(X_1), \quad f_i \nearrow$$

$W \sim$ counter monotone dependence,

$$X_2 = g(X_1), \quad g \searrow$$

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Gaussian and t -copulas \sim elliptic copulas, no closed form

$$C(u_1, \dots, u_n) = F_Z \left(F_1^{-1}(u_1), \dots, F_n^{-1}(u_n) \right)$$

X_i continuous, $F_i(X_i)$ uniform on $]0, 1[$

in general, margins need not be Gaussian (Student)!

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flipping

$$C^-(u, v) = u - C(u, 1 - v)$$

$$(X, Y) \rightarrow (X, -Y)$$

$$C_-(u, v) = v - C(1 - u, v)$$

$$(X, Y) \rightarrow (-X, Y)$$

survival

$$\hat{C}(u, v) = u + v - 1 + C(1 - u, 1 - v)$$

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Ordinal sums

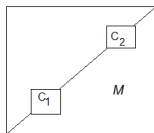


Figure: M -ordinal

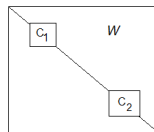


Figure: W -ordinal

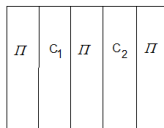


Figure: Π -ordinal

generators $f : [0, 1] \rightarrow [0, \infty]$
strictly decreasing, continuous,

$$f(1) = 0$$

$$C_f(u_1, \dots, u_n) = f^{-1} \left(\min \left\{ f(0), \sum_{i=1}^n f(u_i) \right\} \right)$$

$$g : [-\infty, 0] \rightarrow [0, 1], \quad g(x) = f^{-1} (\min \{f(0), -x\})$$

$$g' \geq 0, \dots, g^{(n-2)} \geq 0, \quad g^{(n-2)} \text{ convex}$$

$$n = 2 \equiv f \text{ convex}$$

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Archimedean copulas

associative and $C_f(u, \dots, u) < u$ for $u \in]0, 1[$

$$f_{[n]}(x) = -\log x \quad \forall n$$

$$f_W(x) = 1 - x \quad \text{only } n = 2$$

n fixed, weakest $C_{f^{[n]}}$

Clayton copula

$$f^{[n]}(x) = 1 - x^{\frac{1}{n-1}}$$

Archimedean copulas

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Clayton

$$f_{\lambda}^C(x) = \frac{x^{-\lambda} - 1}{\lambda}, \quad \lambda \in [-1, \infty[, \quad \lambda \neq 0$$

$$f_0^{\lambda} = f_{\Pi}, \quad \lambda \geq 0 \forall n$$

$\lambda = 1$ ALI-MIKHAIL-HAQ copula

Hamacher product

$$C(u, v) = \frac{uv}{u + v - uv}$$

Gumbel

$$\lambda \in [1, \infty[\quad f_{\lambda}^G(x) = (-\log x)^{\lambda}, \quad \forall n$$

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$$C_{(f)}(u, v) = u f^{-1} \left(\min \left\{ \frac{f(v)}{u}, f(0) \right\} \right)$$

Univariate Conditioning Stable

$$C_{(f_1^c)}(u, v) = \frac{u^2 v}{1 - u + uv}$$

$$C_{(f_{\Pi})}(u, v) = u v^{-\frac{1}{v}}$$

$$C_{(f_W)} = W$$

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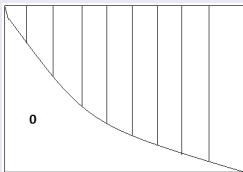
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Associative copula \equiv M -ordinal sums of Archimedean

0 - y - semilinear copulas



$$v = 1 - d(u)$$

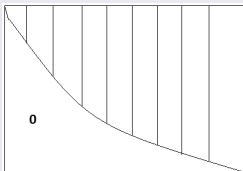
$$C_d(u, v) = u \cdot \max \left\{ 0, 1 + \frac{v-1}{d(u)} \right\}$$

distortion $d : [0, 1] \rightarrow [0, 1]$

$$d(x) \nearrow, \quad \frac{x}{d(x)} \nearrow, \quad d(1) = 1$$

$$C_{id} = W, \quad C_1 = \Pi$$

0 - y - semilinear copulas



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DUCS copulas

Distorted Univariate Conditioning Stable

$$C_{f,d}(u, v) = u \cdot f^{-1} \left(\min \left\{ f(0), \frac{f(v)}{d(u)} \right\} \right) \quad u > 0$$

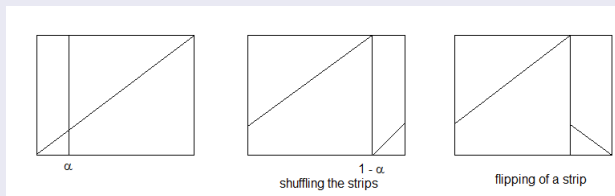
$$d(x) = x^p, \quad p \in]0, 1]$$

$$f(x) = \frac{1}{x} - 1$$

$$C_{f,d}(u, v) = \frac{u^{p+1}v}{1 + u^p v - v}$$

Special singular copulas

Shuffles of M



Each copula can be seen as a limit of shuffles of M

Extreme values (EV) copulas

$$(X_1, Y_1) \sim C$$

$$\vdots$$

$$(X_n, Y_n) \sim C$$

$$(\bigvee X_i, \bigvee Y_i) \sim C_n$$

$$C_n(u, v) = \left(C(u^{1/n}, v^{1/n}) \right)^n \xrightarrow{n \rightarrow \infty} C^*(u, v)$$

$$C^*(u^\lambda, v^\lambda) = (C^*(u, v))^\lambda$$

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$$C^*(u, v) = (uv)^{A\left(\frac{\log u}{\log uv}\right)} = \\ = g^{-1}\left(\min\{g(0), (g(u) + g(v))\} \cdot A\left(\frac{g(u)}{g(u) + g(v)}\right)\right)$$

$$g(x) = -\log x$$

$$A: [0, 1] \rightarrow [0, 1] \quad \text{convex}$$

$$t \vee (1 - t) \leq A(t) \leq 1$$

Archimax

$$C_{f,A}(u, v) = f^{-1}\left(\min\{f(0), (f(u) + f(v))\} \cdot A\left(\frac{f(u)}{f(u) + f(v)}\right)\right)$$



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Thanks for your attention